Surface and Volume Integrals

Surfaces and volumes are the natural domains and boundaries in three-dimensions. In this document we take the techniques of repeated integration[1] and apply them to typical domains in applied mathematics.

Change of Variable

Although we often model systems in the more conventional Cartesian coordinate system[2], it is often more natural to use other coordinate systems such as *polar coordinates*[3] in two dimensions, cylindrical polar coordinates[4] or *spherical polar coordinates*[5] in three dimensions. Hence we often need to transform our equations between one coordinate system and another, requiring a change of variable. Also, as is the case in one-variable integration[6], a change of variable can make an integral easier to evaluate.

Two Dimensions

First in two dimensions, let the normal Cartesian coordinates *x* and *y* be defined in terms of alternative variables *u* and *v*; *x=x(u,v)*, *y=y(u,v)*, then

$$\iint_R f(x,y)dx\,dy = \iint_{R'} f(x(u,v),y(u,v))|J|du\,dv\ ,$$

where *R'* is the domain in *u,v* coordinates which is equivalent to the domain *R* in Cartesian coordinates and *J* is the *Jacobian* of the transformation.

The Jacobian is defined as follows:

$$\mathrm{J}=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(Note the |*| denotes the determinant[7] of the matrix and the ∂ denotes partial derivatives[8].)

Three Dimensions

In three dimensions, let the normal Cartesian coordinates *x,y* and *z* be defined in terms of alternative variables *u,v* and *w*; *x=x(u,v,w)*, *y=y(u,v,w)*, and *z=z(u,v,w)* then

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[1] Repeated Integration
[2] Cartesian Coordinate System
[3] Polar Coordinate System
[4] Cylindrical Polar Coordinates
[5] Spherical Polar Coordinates
[6] Integration
[7] Inverse of a 2x2 Matrix
[8] Partial Differentiation

$$\iiint_R f(x,y,z)\,dx\,dy\,dz = \iiint_{R'} f\big(x(u,v,w),y(u,v,w),z(u,v,w)\big)\,|J|du\,dv\,dw,$$

where R' is the domain in u,v coordinates which is equivalent to the domain R in Cartesian coordinates and

$$\boldsymbol{J} = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

(Note the |*| denotes the determinant[9] of the matrix.)

2D Cartesian to Polar transformation

In the polar coordinate system[10] the Cartesian coordinates are written in terms of different variables; a radial and angular coordinate: $x = r\cos(\theta), y = r\sin(\theta)$.

The Jacobian for the change of variable from Cartesian to polar coordinates is given by

$$\mathrm{J} = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r\,.$$

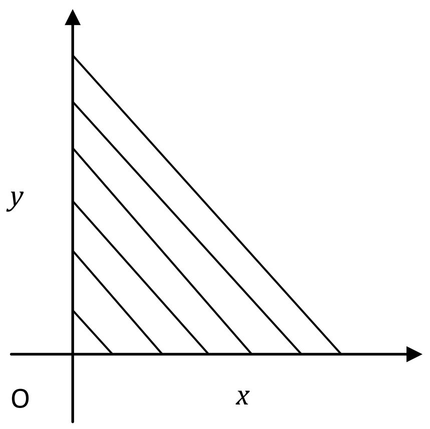
Hence

$$\iint_R f(x,y)dx\,dy = \iint_{R'} f(r\cos(\theta), r\sin(\theta)\; r\,dr\,d\theta\;.$$

<u>Example</u>

Find $\int_0^\infty \int_0^\infty \frac{1}{(x^2+y^2+1)^2}\,dx\,dy.$

First note the domain of integration:



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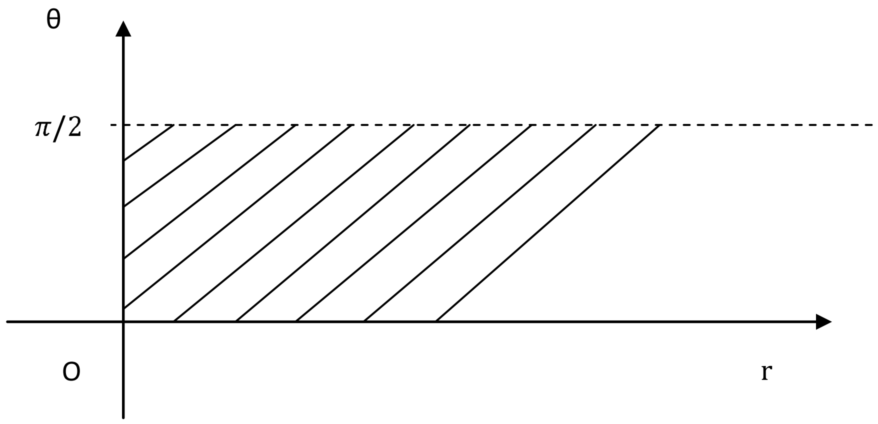
9 [Inverse of a 3x3 Matrix](#)
10 [Polar Coordinate System](#)

First we make the substitution $x = r\cos(\theta), y = r\sin(\theta)$ and reconfigure the domain of integration in terms of r and θ.

$$\int_0^{\pi/2}\int_0^{\infty}\frac{1}{(x^2+y^2+1)^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{1}{((r\cos(\theta))^2+(r\sin(\theta))^2+1)^2}\,r\,dr\,d\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\infty}\frac{r}{(r^2+1)^2}\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[\frac{-1/2}{(r^2+1)}\right]_0^{\infty}\,d\theta = \int_0^{\frac{\pi}{2}}1/2\;d\theta = \frac{\pi}{4}$$

The domain of integration in terms of r and θ is illustrated in the following graph.



3D Cartesian to Cylindrical Polar Coordinates

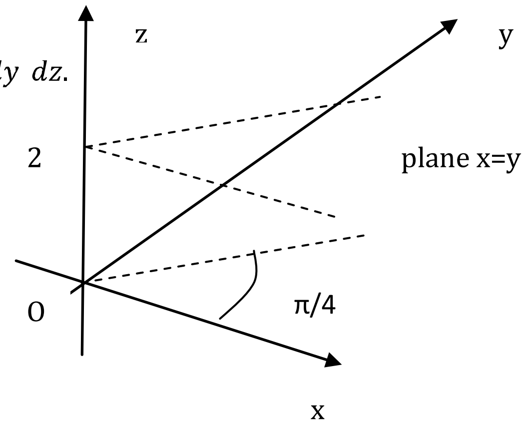
In the cylindrical polar coordinate system[11] the Cartesian coordinates are written in terms of different variables: $x = R\cos(\theta), y = R\sin(\theta), z = z$.

$$\boldsymbol{J} = \frac{\partial(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})}{\partial(\boldsymbol{R},\boldsymbol{\varphi},\boldsymbol{z})} = \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -R\sin(\theta) & 0 \\ \sin(\theta) & R\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = R$$

[11] Cylindrical Polar Coordinate System

Example

Find $\int_0^2 \int_0^\infty \int_0^y \exp(-x^2 - y^2 + z)\ dx\, dy\ dz$.



Note the domain of integration; it is bounded by the planes z=0, z=2, y=0 and x=y.

In cylindrical coordinates we note that R ranges from 0 to ∞, θ ranges from 0 to π/4 and z ranges from 0 to 2. Hence we can rewrite the integral in the form:

$$\int_0^2 \int_0^\infty \int_0^y \exp(-x^2 - y^2 + z)\ dx\, dy\ dz$$

$$= \int_0^2 \int_0^{\frac{\pi}{4}} \int_0^\infty \exp(-(R\cos(\theta))^2 - (R\sin(\theta))^2 + z)\ R\, dR\, d\theta\ dz$$

$$= \int_0^2 \int_0^{\frac{\pi}{4}} \int_0^\infty \exp(-R^2 + z)\ R\, dR\, d\theta\ dz = \int_0^2 \int_0^{\frac{\pi}{4}} [-\exp(-R^2 + z)]_0^\infty\ \ d\theta\ dz$$

$$= \int_0^2 \int_0^{\frac{\pi}{4}} \exp(z)\ \ d\theta\ dz = \frac{\pi}{4}\int_0^2 \exp(z) dz = \frac{\pi}{4}[\exp\,(z)]_o^2 = \frac{\pi}{4}(\exp(2) - 1)\,.$$

written in terms of different variables: $x = r\sin(\theta)\cos(\varphi), y = r\sin(\theta)\sin(\varphi)\,, z = r\cos\,(\theta)$.

$$\boldsymbol{J} = \frac{\boldsymbol{\partial(x,y,z)}}{\boldsymbol{\partial(R,\varphi,z)}} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \sin\,(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{vmatrix} =$$

$$= \cos(\theta)\,\{r\cos(\theta)\cos(\varphi) r\sin\,(\theta)\cos(\varphi)\ -\ -r\sin(\theta)\sin(\varphi)\, r\cos(\theta)\sin(\varphi)\} -$$

$$- r\sin(\theta)\,\{\sin(\theta)\cos(\varphi) r\sin(\theta)\cos(\varphi) - - r\sin(\theta)\sin(\varphi)\sin(\theta)\sin(\varphi)\}$$

$$= \cos(\theta)\,\{r^2\cos(\theta)\sin(\theta)\} + r\sin(\theta)\,\{r\sin^2(\theta)\} = r^2\sin(\theta)$$

<u>Example</u>

Find

$$\iiint_R \frac{1}{(x^2+y^2+z^2)}\,dx\,dy\,dz\,,$$

where R represents a sphere of radius 3.

<u>Answer</u>

Substituting $x = r\sin(\theta)\cos(\varphi), y = r\sin(\theta)\sin(\varphi)\,,\ z = r\cos(\theta)$.

$$\iiint_R \frac{1}{(x^2+y^2+z^2)}\,dx\,dy\,dz$$

$$= \int_0^3\int_0^{2\pi}\int_0^{\pi} \frac{1}{\left(\left(r\sin(\theta)\cos(\varphi)\right)^2 + (r\sin(\theta)\sin(\varphi))^2 + (r\cos(\theta))^2\right)}\,r^2\sin(\theta)\ d\theta\,d\varphi\,dR$$

$$= \int_0^3\int_0^{2\pi}\int_0^{\pi}\sin(\theta)\ d\theta\,d\varphi\,dR == \int_0^3\int_0^{2\pi}[-\cos(\theta)]_0^{\pi}\,d\varphi\,dR = 12\pi\,.$$